Chapter 7.

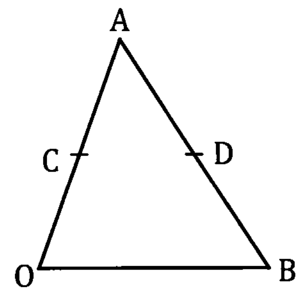
Proofs using vectors.

A vector approach can be used to prove certain geometrical facts, as the next example demonstrates. When using such an approach our accepted facts or axioms include the basic ideas that follow from our understanding of vectors, and the results that follow from these basic ideas. For example:

- Equal vectors have the same magnitude and the same direction.
- If $\mathbf{a} = \lambda\mathbf{b}$ then if $\lambda > 0$ **a** and **b** are like parallel vectors
  and if $\lambda < 0$ **a** and **b** are unlike parallel vectors.
- Vectors can be added (or subtracted) using a triangle of vectors or the parallelogram law.
- If $h\mathbf{a} = k\mathbf{b}$ then either **a** and **b** are parallel vectors
  or $h = k = 0$.

Example 1

To prove: **The line from the mid-point of one side of a triangle to the mid-point of a second side is parallel to, and half as long as, the third side.**



Consider triangle OAB with C the mid-point of OA and D the mid-point of AB.

We have to prove that $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{OB}$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

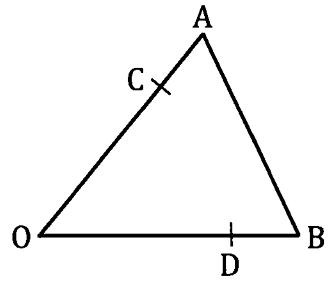
$$
\begin{aligned}
\text{Now } \overrightarrow{CD} &= \overrightarrow{CA} + \overrightarrow{AD} \\
&= \tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} \qquad \text{(C and D are mid points.)} \\
&= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}(-\mathbf{a} + \mathbf{b}) \\
&= \tfrac{1}{2}\mathbf{b} \\
&= \tfrac{1}{2}\overrightarrow{OB} \quad \text{as required.}
\end{aligned}
$$

Thus the line from the mid-point of one side of a triangle to the mid-point of a second side is parallel to, and half as long as, the third side.

Exercise 7A

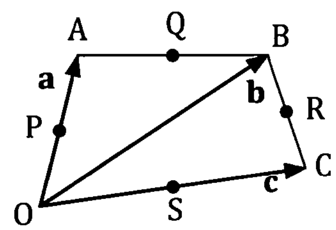
1. To prove: **The line drawn from the point that divides one side of a triangle in a certain ratio, to the point that divides a second side in the same ratio is parallel to the third side.**

In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$, and $\overrightarrow{OB} = \mathbf{b}$.
C and D are points on OA and OB respectively such that $\overrightarrow{OC} = h\overrightarrow{OA}$ and $\overrightarrow{OD} = h\overrightarrow{OB}$.
Prove that CD is parallel to AB.



2. To prove: **The mid-points of the sides of a quadrilateral form a parallelogram.**

Consider the quadrilateral OABC with $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.
P, Q, R and S are the mid-points of OA, AB, BC and OC respectively. Find vector expressions for each of $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{SR}$ and $\overrightarrow{PS}$ and hence prove that the mid-points of the sides of a quadrilateral form a parallelogram.



3. To prove: **The diagonals of a parallelogram bisect each other.** (Method 1.)

OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$, and $\overrightarrow{OC} = \mathbf{c}$.

M is the mid-point of the diagonal OB.

Find $\overrightarrow{CM}$ and $\overrightarrow{CA}$ in terms of **a** and **b** and hence show that M lies on CA and is the mid-point of CA.

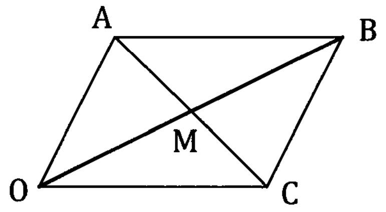
4. To prove: **The diagonals of a parallelogram bisect each other.** (Method 2.)

OABC is a parallelogram with

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

The diagonals OB and AC meet at M.

If $\overrightarrow{AM} = h\overrightarrow{AC}$ and $\overrightarrow{OM} = k\overrightarrow{OB}$ use the fact that $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$ to show that $h = k = \frac{1}{2}$.



5. (a) To prove: **If a quadrilateral is such that its diagonals bisect each other then the quadrilateral is a parallelogram.**

OABC is a quadrilateral. If M is the mid-point of both diagonals prove that

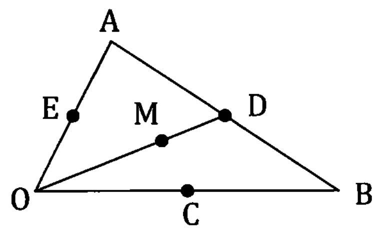
$$\overrightarrow{AB} = \overrightarrow{OC} \text{ and } \overrightarrow{OA} = \overrightarrow{CB}.$$

i.e. prove that the quadrilateral is a parallelogram.

(b) Combine the result proved in this question with the result of the previous question into one statement using the symbol $\Leftrightarrow$ and also write the statement using the "if and only if" statement. (See page 17 if need be.)

6. To prove: **The medians of a triangle intersect at a point two thirds of the way along their length measured from the vertex.** (Method 1.)

In triangle OAB, C, D and E are the mid-points of OB, AB and OA respectively. M is a point on OD such that $\overrightarrow{OM} = \frac{2}{3}\overrightarrow{OD}$. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(a) Find the following in terms of **a** and/or **b**.

(i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AC}$ (iii) $\overrightarrow{AD}$

(iv) $\overrightarrow{OD}$ (v) $\overrightarrow{OM}$ (vi) $\overrightarrow{AM}$

(b) Prove that M lies on AC and is such that AM : MC = 2 : 1.

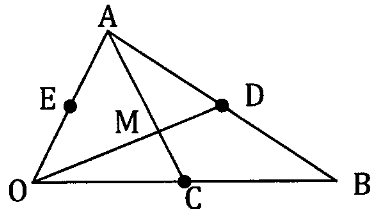
(c) Prove that M lies on BE and is such that BM : ME = 2 : 1.

7. To prove: **In a triangle, if a line is drawn from a point that divides one side in a given ratio, parallel to a second side, then it divides the third side in the same ratio.**

In triangle ABC, $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$. D is a point on AB such that $\overrightarrow{AD} = h\overrightarrow{AB}$. A line through D, parallel to AC, cuts CB at point E. Prove that $\overrightarrow{CE} = h\overrightarrow{CB}$.

(Hint: Let $\overrightarrow{CE} = k\overrightarrow{CB}$ and then prove $k = h$.)

8. To prove: **The medians of a triangle intersect at a point two thirds of the way along their length measured from the vertex.** (Method 2.)



In triangle OAB, C, D and E are the mid-points of OB, AB and OA respectively. OD and AC meet at M.

$\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OM} = h\overrightarrow{OD}$ and $\overrightarrow{AM} = k\overrightarrow{AC}$.

Use the fact that $\overrightarrow{OA} + \overrightarrow{AM} = \overrightarrow{OM}$ to determine h and k.

Show that M also lies on BE and, if $\overrightarrow{BM} = \lambda\overrightarrow{BE}$, find λ.

9. In the quadrilateral OABC, X and Y are the mid-points of the diagonals OB and AC respectively. Prove that $\overrightarrow{OA} + \overrightarrow{BA} + \overrightarrow{OC} + \overrightarrow{BC} = 4\overrightarrow{XY}$.

10. In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and C is the mid-point of AB. D and E are points on OA and OB respectively and DE cuts OC at F.

$\overrightarrow{OD} = h\overrightarrow{OA}$, $\overrightarrow{OE} = k\overrightarrow{OB}$ and $\overrightarrow{OF} = m\overrightarrow{OC}$.

(a) Express $\overrightarrow{DF}$ in terms of h, m, **a**, and **b**.

(b) Express $\overrightarrow{FE}$ in terms of k, m, **a**, and **b**.

If $\overrightarrow{DF} = \overrightarrow{FE}$ prove that (c) $h = k = m$. (d) DE is parallel to AB.

Miscellaneous Exercise Seven.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Discuss the correctness or otherwise of each of the following "if and only if" statements.
   (a) A triangle is scalene if and only if it has three different length sides.
   (b) A positive whole number is a multiple of 5 if and only if it ends with a zero.

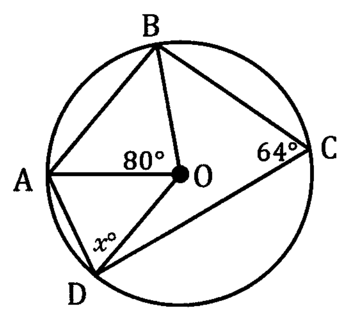
2. In how many ways can the three positions of Chairman, Secretary and Treasurer be chosen from a committee of 8 people if each position must be held by a different person?

3. (a) How many different subcommittees of six people could be selected from a full committee of 15 people?
   (b) In how many ways can a particular subcommittee of six be arranged in a line for a photograph?

4. How many different 6 letter arrangements can be made each consisting of 6 different letters of the alphabet, with exactly one of the 6 being a vowel?

5. In the diagram on the right points A, B, C and D lie on a circle centre O.

   Given that $\angle BOA = 80°$,
   $\angle BCD = 64°$
   and $\angle ADO = x°$
   prove that $x = 66$.



6. Points A and B have position vectors $\mathbf{i} + 5\mathbf{j}$ and $7\mathbf{i} - \mathbf{j}$ respectively.

   Find the position vector of (a) point P, on AB, such that $\overrightarrow{AP} : \overrightarrow{PB} = 4 : 1$,

   (b) point Q on AB produced, such that $\overrightarrow{AQ} : \overrightarrow{QB} = 4 : -1$.

7. Point A has position vector $-2\mathbf{i} + 7\mathbf{j}$.
   Relative to point A a second point, B, has position vector $8\mathbf{i} + 3\mathbf{j}$.
   I.e. ${}_B\mathbf{r}_A = 8\mathbf{i} + 3\mathbf{j}$. All units are in metres.
   When timing commences an object moving with constant velocity of

   $$(3\mathbf{i} - 2\mathbf{j}) \text{ m/sec}$$

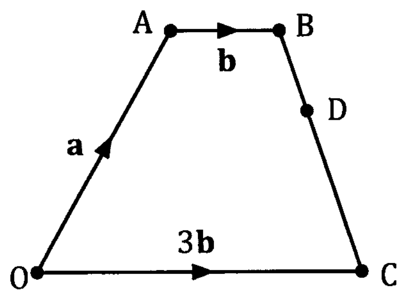
   is at point B. *Exactly* how far is this object from the origin 2 seconds later?

8. Vectors **a**, **b** and **c** are such that
$\mathbf{b} = -7\mathbf{i} + 24\mathbf{j}$,
$\mathbf{c} = 3\mathbf{i} - 4\mathbf{j}$,
a and **b** have the same magnitude,
a and **c** have exactly the same direction.
Find the exact magnitude of $(\mathbf{a} + \mathbf{b})$.

9. Airfield B is 600 km south-east of airfield A.
An aeroplane that can fly at 300 km/h in still air is to make the journey from A to B with a wind of 40 km/h blowing from the west.

If the wind remains the same throughout determine the time taken (to the nearest minute) for the aircraft to fly from (a) A to B, (b) B to A.

10. In the diagram on the right $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{OC} = 3\mathbf{b}$.
D is a point on BC such that BD : DC = 1 : 2.
Express each of the following vectors in terms of **a** and **b**.



(a) $\overrightarrow{BC}$, (b) $\overrightarrow{BD}$, (c) $\overrightarrow{OD}$.
Now suppose that OD continued meets AB continued at E and that:

$$\overrightarrow{BE} = h\mathbf{b} \text{ and } \overrightarrow{DE} = k\overrightarrow{OD}.$$

Find h and k.

11. A company wishes to give each of the products it sells a code using the letters of the alphabet, i.e. A, B, C, ... Z. In each code the order of the letters is significant. Thus whilst one product might have code ABCD, the code ABDC is different and would indicate a different product.
How many different codes are possible if each code consists of
(a) 4 different letters,
(b) 4 letters with multiple use of letters in a code permitted,
(c) 6 different letters,
(d) 6 letters with multiple use of letters in a code permitted.
If the company wants
- all the codes to have the same number of letters,
- all the codes to start with the letters AR, in that order,
- no code to feature a letter more than once,
- to have at least 12 500 different codes.

What is the least number of letters it should have in each code?